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The base-flow and near-wake problem at very low Reynolds numbers

Part 1. The Stokes approximation

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The general solutions of the Stokes approximate equations of motion are derived for two-dimensional and axisymmetric flows in the half-space $x > 0$, for an arbitrarily given velocity field in the plane $x = 0$. There is assumed to be no solid surface in the half-space. According to whether the velocity at infinity is zero or not, the solutions can be said to describe either jet-type or wake-type flows. Only the latter category is considered; numerical examples are worked out and properties of the base flow at very low Reynolds numbers are investigated. A recirculating flow region may exist, but the flow properties are not sensitive to this feature.

1. Introduction

The base-flow and near-wake problems have attracted much attention in the last few years, especially in connexion with hypersonic flight and atmospheric re-entry. Most of the effort has been directed towards explaining the base-flow structure and properties for the case of high speed, high Reynolds number, laminar flows.

The characteristic parameter which determines the importance and nature of the viscous effects in the base region is the ratio $\delta = \delta_1/a_1$ of initial boundary-layer thickness to base height or base radius a_1 (figure 1). For small values of δ (large-Reynolds-number case) the viscous effects are considered to be non-negligible only in a very thin, so-called mixing layer made up of the boundary layer and of an entrained layer in the recirculating flow; the thickness of this layer grows with δ so that a greater and greater part of the recirculating flow becomes viscous, until the concept of a mixing layer becomes invalid for δ large enough; it seems likely that δ need not be very large (less than one) for this to happen. If L_1 is the body length and $R_{L_1} = UL_1/\nu$, δ is of the order of $L_1/(a_1\sqrt{R_{L_1}})$† and hence is directly proportional to L_1/a_1 for a fixed R_{L_1}; it is clear that for a given R_{L_1} the more slender the body the more important are viscous effects in the base flow.

The present work is concerned with the structure of the base flow and the near-wake flow for values of δ large enough for this flow to be entirely viscous, this case being of interest for slender bodies at low enough Reynolds numbers. We shall restrict ourselves to the case of an incompressible fluid. The base of the body is assumed plane; the base-flow region of interest is then the half-space downstream of the base flow.

† Assuming that R_{L_1} is large enough for boundary-layer concepts to apply.

At this point, the problem is to solve the incompressible Navier–Stokes equations in the half-space $x > 0$, with the necessary boundary conditions given at $x = 0$, the base plane, and with the condition of uniform flow at infinity. As has been indicated, the case we shall consider is one where viscous forces are important in the entire base region; furthermore, in the vicinity of the base, because of the no-slip condition, they should far outweigh inertia forces. This indicates that the Stokes approximation, in which inertia terms are neglected, should be applicable, at least locally. It is difficult to determine *a priori* the extent of this region for given δ and R_{L_1}, although we know it to be inversely proportional to a characteristic Reynolds number such as Ua_1/ν; but this point can and will be examined *a posteriori* from the solution.

We therefore propose first to treat the base-flow problem by means of the Stokes approximate equations of motion (Part 1), despite the fact that we shall use boundary conditions at $x = 0$ far from the body, in a region where inertia effects are dominant. To obtain at least a qualitative estimate of the effect of the inertia terms, Oseen's form of the equations of motion will also be considered (Part 2).

2. The basic equations

The co-ordinate system is shown in figure 1; the x_1-axis is taken perpendicular to the base plane, with origin at the base centre, and is therefore the axis of symmetry when there is one.† In axisymmetric flow cylindrical co-ordinates are used, y_1 being the radial distance (in two-dimensional flow y_1 is the lateral distance).

Dimensionless dependent and independent variables are introduced by the following definitions:

$$\begin{aligned} x &= x_1/a_1, & y &= y_1/a_1, \\ u &= u_1/U, & v &= v_1/U, \\ p &= p_1/\rho U^2, & \bar{p} &= p_1 a_1/\mu U = pR_{a_1}, \\ \Omega &= \Omega_1 a_1/U, & \psi &= \psi_1/U a_1^{j+1}, \end{aligned}$$

where u_1 and v_1 are the velocity components parallel to the x_1-, y_1-directions, respectively; p_1 is the pressure; ρ the density; U the x_1-component of free-stream velocity; Ω_1 the vorticity; and ψ_1 the stream function. Also,

$$\begin{aligned} j &= 0 \quad \text{in two-dimensional flow,} \\ j &= 1 \quad \text{in axisymmetric flow.} \end{aligned}$$

In terms of the non-dimensional variables defined above the incompressible Navier–Stokes equations may be written

$$\frac{\partial}{\partial x}(y^j u) + \frac{\partial}{\partial y}(y^j v) = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = -\frac{1}{R_{a_1}} y^{-j}\frac{\partial}{\partial y}(y^j\Omega), \tag{2.2a}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{R_{a_1}}\frac{\partial \Omega}{\partial x}. \tag{2.2b}$$

† We do not necessarily assume symmetry of the flow in the two-dimensional case.

The vorticity is
$$\Omega = -\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}, \tag{2.3}$$
and satisfies the equation
$$u\frac{\partial \Omega}{\partial x}+v\frac{\partial \Omega}{\partial y}-j\frac{v}{y}\Omega = \frac{1}{R_{a_1}}\left\{\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial y^2}+j\left(\frac{1}{y}\frac{\partial \Omega}{\partial y}-\frac{\Omega}{y^2}\right)\right\}. \tag{2.4}$$

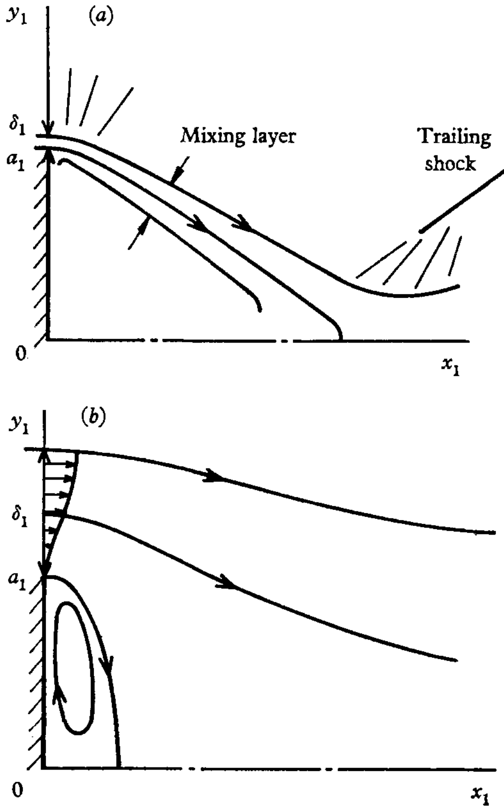

FIGURE 1. Sketch of base flow. (*a*) High-Reynolds-number, supersonic flow; (*b*) low-Reynolds-number flow.

The continuity equation (2.1) is satisfied by introducing a stream-function $\psi(x,y)$ such that
$$u = y^{-j}\frac{\partial \psi}{\partial y}, \quad v = -y^{-j}\frac{\partial \psi}{\partial x}. \tag{2.5}$$
Introducing these relations into equation (2.3),
$$\Omega = -y^{-j}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\frac{j}{y}\frac{\partial \psi}{\partial y}\right). \tag{2.6}$$

3. Method of solution

If inertia terms are neglected, the momentum equations (2.2) become
$$\frac{\partial \overline{p}}{\partial x} = -y^{-j}\frac{\partial}{\partial y}(y^j\Omega), \tag{3.1a}$$
$$\frac{\partial \overline{p}}{\partial y} = \frac{\partial \Omega}{\partial x}, \tag{3.1b}$$

and, as a result, the vorticity equation (2.4) becomes

$$\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial y^2}+j\left(\frac{1}{y}\frac{\partial\Omega}{\partial y}-\frac{\Omega}{y^2}\right)=0. \tag{3.2}$$

Substituting the right-hand side of equation (2.6) for the vorticity in equation (3.2), we obtain the following well known equation for the stream function in Stokes flow

$$\nabla_j^4\psi=0, \tag{3.3}$$

where

$$\nabla_j^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{j}{y}\frac{\partial}{\partial y}.$$

We shall also use the following notations:

if $j=0$, $\nabla_0^2\equiv\nabla^2$, the Laplacian in two dimensions,

if $j=1$, $\nabla_1^2\equiv D^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{y}\frac{\partial}{\partial y}$.

The method followed consists in solving for the stream function first, equation (3.3), and then for the pressure, equations (3.1). The boundary conditions on ψ are

$$\psi(0,y)=\psi_i(y), \tag{3.4a}$$

$$\frac{\partial\psi}{\partial x}(0,y)=-y^jv_i(y). \tag{3.4b}$$

In axisymmetric flow, y takes on only positive (or zero) values and we have the additional boundary condition

$$\psi(x,0)=0. \tag{3.4c}$$

Note that from equation (3.4*a*) we deduce

$$u(0,y)=u_i(y)=y^{-j}\frac{d\psi_i(y)}{dy}. \tag{3.4d}$$

The condition for uniform flow at infinity is†

$$\psi(x,y)\sim\frac{y^{j+1}}{j+1}-\frac{V}{U}x \quad \text{as} \quad (x^2+y^2)\to\infty.$$

This condition is taken care of by equations (3.4*a*) and (3.4*b*), and is not actually needed to determine the solution.

Equation (3.3) is only a particular case of a more general class of equations which have been studied by Almansi (1899), Payne (1958), and Weinstein (1955) among others, and which have remarkable properties regarding possible decompositions of the solutions. Payne & Pell (1960) pointed out the usefulness of such decompositions in connexion with axially symmetric Stokes-flow problems. The property of which we shall make use is the following:

The general solution of the equation

$$\nabla_j^{2n}\psi=0 \tag{3.5}$$

† V is non zero only in the case of two-dimensional asymmetric flow.

in the domain $x > x_0$, can be written in the form

$$\psi = \sum_{k=1}^{n} x^{k-1} V_k, \tag{3.6}$$

where all the functions V_k are solutions of

$$\nabla_j^2 V_k = 0. \tag{3.7}$$

It is easily verified that each term of the decomposition (3.6) is a particular solution of equation (3.5). Therefore, one has to solve partial differential equations of second order instead of order $2n$.

Making use of this representation for the solution of equation (3.3), with $n = 2$, we can write

$$\psi = V_1 + x\overline{V}_2, \tag{3.8}$$

where

$$\nabla_j^2 V_1 = \nabla_j^2 \overline{V}_2 = 0. \tag{3.9}$$

The boundary conditions (3.4*a*) and (3.4*b*) result in

$$V_1(0, y) = \psi_i(y), \tag{3.10}$$

$$\left.\frac{\partial V_1}{\partial x}\right|_{x=0} + \overline{V}_2(0, y) = -y^j v_i(y). \tag{3.11}$$

Let us replace $\overline{V}_2$ by a new function V_2, defined by

$$V_2 = -\{(\partial V_1/\partial x) + \overline{V}_2\},$$

so that ψ becomes

$$\psi = V_1 - x\{(\partial V_1/\partial x) + V_2\}, \tag{3.12}$$

and equation (3.11) gives

$$V_2(0, y) = y^j v_i(y). \tag{3.13}$$

From the definition of V_2, and from equations (3.9), it is clear that

$$\nabla_j^2 V_2 = 0, \tag{3.14a}$$

together with

$$\nabla_j^2 V_1 = 0. \tag{3.14b}$$

In axisymmetric flow, the additional boundary condition (3.4*c*) gives

$$V_1(x, 0) = V_2(x, 0) = 0. \tag{3.15}$$

In terms of V_1 and V_2, the vorticity, equation (2.6), can be written

$$\Omega = 2y^{-j} \frac{\partial}{\partial x}\{(\partial V_1/\partial x) + V_2\}. \tag{3.16}$$

Making use of (3.16), equations (3.1) for the pressure are easily integrated with the result

$$\overline{p} - \overline{p}_\infty = -2y^{-j} \frac{\partial}{\partial y}\{(\partial V_1/\partial x) + V_2\}. \tag{3.17}$$

Therefore, the problem is reduced to finding the solutions of equations (3.14) with the boundary conditions (3.10), (3.13), and, if $j = 1$, (3.15). This is done in §§4 and 5 for two-dimensional and axisymmetric flows, respectively.

4. Two-dimensional Stokes flow

4.1. *The stream function*

For $j = 0$, equations (3.14) are Laplace's equations, to be solved for $x > 0$

$$\nabla^2 V_1 = \nabla^2 V_2 = 0, \tag{4.1}$$

with the boundary conditions (3.10) and (3.13)

$$V_1(0, y) = \psi_i(y), \tag{4.2a}$$

$$V_2(0, y) = v_i(y). \tag{4.2b}$$

V_1 and V_2 are obtained immediately by using Poisson's integral for a half-plane (Courant & Hilbert 1962, vol. II, p. 268):

$$V_1(x, y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x}{x^2+(\eta-y)^2}\psi_i(\eta)\,d\eta, \tag{4.3a}$$

$$V_2(x, y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x}{x^2+(\eta-y)^2}v_i(\eta)\,d\eta. \tag{4.3b}$$

Since $\psi_i(\eta) \to \infty$ as $\eta \to \infty$, the use of Poisson's integral to obtain the solution represented by equation (4.3a) would seem to be invalid. However,

$$\psi_i(\eta) \sim \eta + \text{bounded terms as} \quad \eta \to \infty,$$

and the unbounded part in $\psi_i(\eta)$, namely η, contributes the finite term y to V_1 since

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2+(\eta-y)^2}\eta\,d\eta = y.$$

(A formal justification for the use of Poisson's integral may be accomplished by introducing the new function $\overline{V}_1(x, y) = V_1(x, y) - y$; $\overline{V}_1$ is then a harmonic function which is bounded at $x = 0$ for all y, including $y \to \infty$. If we apply Poisson's integral to solve for $\overline{V}_1$ and then transform back to V_1 we obtain exactly equation (4.3a).) Finally, we get

$$\psi = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{x^3}{[x^2+(\eta-y)^2]^2}\psi_i(\eta)\,d\eta - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{x^2+(\eta-y)^2}v_i(\eta)\,d\eta. \tag{4.4}$$

4.2. *Velocity field*

We can obtain u from equation (4.4) by calculating $\partial\psi/\partial y$; one can also note that $\nabla^4 u = 0$, and

$$u(0, y) = u_i(y),$$

$$\frac{\partial u}{\partial x}(0, y) = -\frac{\partial v}{\partial y}(0, y) = -\frac{dv_i}{dy},$$

so that, by the same same method as that used to solve for ψ, we get

$$u = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{x^3}{[x^2+(\eta-y)^2]^2}u_i(\eta)\,d\eta - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{x^2+(\eta-y)^2}\frac{dv_i(\eta)}{d\eta}\,d\eta. \tag{4.5}$$

v is given by equations (2.5) and (4.4) as

$$v = -x\frac{\partial^2 V_1}{\partial y^2} + V_2 + x\frac{\partial V_2}{\partial x}.$$

Hence

$$v = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x^2}{x^2+(\eta-y)^2}\frac{du_i(\eta)}{d\eta}d\eta + \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{x(\eta-y)^2}{[x^2+(\eta-y)^2]^2}v_i(\eta)\,d\eta. \tag{4.6}$$

The case of symmetric flow is of particular interest; then $\psi_i(\eta)$ and $v_i(\eta)$ are odd functions of η, while $u_i(\eta)$ and $dv_i/d\eta$ are even in η; taking advantage of this fact, the velocity on the axis of symmetry can be written in the form

$$u_0(x) = \frac{4}{\pi}\int_0^{\infty}\frac{x^3}{(x^2+\eta^2)^2}u_i(\eta)\,d\eta - \frac{2}{\pi}\int_0^{\infty}\frac{x^2}{x^2+\eta^2}\frac{dv_i(\eta)}{d\eta}d\eta. \tag{4.7}$$

4.3. *Pressure field*

Equations (3.17) and (4.3) yield

$$\begin{aligned}\bar{p}-\bar{p}_\infty &= -2\frac{\partial}{\partial y}(V_2+\partial V_1/\partial x)\\ &= -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{(\eta-y)^2-x^2}{[x^2+(\eta-y)^2]^2}u_i(\eta)\,d\eta - \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{x}{x^2+(\eta-y)^2}\frac{dv_i(\eta)}{d\eta}d\eta. \end{aligned} \tag{4.8}$$

In the case of symmetric flow, the pressure on the axis of symmetry becomes

$$\bar{p}_0(x)-\bar{p}_\infty = -\frac{4}{\pi}\int_0^{\infty}\frac{\eta^2-x^2}{(\eta^2+x^2)^2}u_i(\eta)\,d\eta - \frac{4}{\pi}\int_0^{\infty}\frac{x}{\eta^2+x^2}\frac{dv_i(\eta)}{d\eta}d\eta. \tag{4.9}$$

Another interesting result is the base pressure; in general the values taken at $x = 0$ by various quantities such as $\partial v/\partial x$, p, Ω, etc., which involve partial derivatives of V_1 or V_2 of odd order in x, cannot be inferred directly from the solutions given above for V_1 and V_2 because these integral representations are singular at $x = 0$; in other words, $(\partial V_1/\partial x)|_{x=0}$ and $(\partial V_2/\partial x)|_{x=0}$ are not in general† directly obtainable from equations (4.3). However, in the base-flow problem the integration is actually taken over $|\eta| > 1$ since ψ_i and v_i are zero‡ for $|\eta| < 1$ (i.e. along the base), so that for $|y| < 1$ the integral representation is not singular at $x = 0$; therefore we use equation (4.8) to obtain the base pressure $\bar{p}_b(y)$ $(x=0, |y| < 1)$ as

$$\bar{p}_b(y)-\bar{p}_\infty = \frac{2}{\pi}\left(\int_{-\infty}^{-1}+\int_1^{\infty}\right)\frac{1}{y-\eta}\frac{du_i(\eta)}{d\eta}d\eta,$$

and, in the case of symmetric flow,

$$\bar{p}_b(y)-\bar{p}_\infty = -\frac{4}{\pi}\int_1^{\infty}\frac{\eta}{\eta^2-y^2}\frac{du_i(\eta)}{d\eta}, \quad |y| < 1. \tag{4.10}$$

† Unless, of course, the integrations involved can be performed analytically.

‡ Except in the cases of base bleeding and slip at the wall.

5. Axisymmetric Stokes flow

5.1. *The stream function*

For $j = 1$, equations (3.14) become

$$D^2V_1 = D^2V_2 = 0, \tag{5.1}$$

where $$D^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{y}\frac{\partial}{\partial y} \quad \text{and} \quad y \geqslant 0.$$

The boundary conditions (3.10), (3.13), and (3.15) are

$$V_1(0,y) = \psi_i(y), \tag{5.2a}$$

$$V_2(0,y) = yv_i(y), \tag{5.2b}$$

$$V_1(x,0) = V_2(x,0) = 0. \tag{5.2c}$$

The general properties of equation (5.1) have been studied by Brousse (1956), who gave several solutions, in particular the solution of the boundary-value problems for a half-circle with limiting diameter on Ox, and for the quarter-plane $x > 0, y > 0$.

In this latter case, and, if the boundary value is zero on the x-axis, the solution can be written in the form

$$V(x,y) = \frac{3}{\pi}y^2\int_{\eta=0}^{\infty}\int_{\theta=0}^{\pi}\frac{x\sin^2\theta}{(MN^2)^{\frac{5}{2}}}\,\eta V_i(\eta)\,d\theta\,d\eta, \tag{5.3}$$

where $$MN^2 = x^2+y^2+\eta^2-2y\eta\cos\theta,$$

and $$V_i(y) = V(0,y).$$

In Appendix I, we give a different, and we believe, new method for arriving at equation (5.3) by using Poisson's integral formula. Therefore V_1 and V_2 are obtained immediately in the form given by equation (5.3), with $V_i(\eta)$ respectively equal to $\psi_i(\eta)$ and $\eta v_i(\eta)$. For reference purposes, we write them down

$$V_1(x,y) = \frac{3}{\pi}y^2\int_{\eta=0}^{\infty}\int_{\theta=0}^{\pi}\frac{x\sin^2\theta}{(MN^2)^{\frac{5}{2}}}\,\eta\psi_i(\eta)\,d\theta\,d\eta, \tag{5.4a}$$

$$V_2(x,y) = \frac{3}{\pi}y^2\int_{\eta=0}^{\infty}\int_{\theta=0}^{\pi}\frac{x\sin^2\theta}{(MN^2)^{\frac{5}{2}}}\,\eta^2 v_i(\eta)\,d\theta\,d\eta. \tag{5.4b}$$

We also find that

$$V_1 - x\frac{\partial V_1}{\partial x} = \frac{15}{\pi}y^2\int_{\eta=0}^{\infty}\int_{\theta=0}^{\pi}\frac{x^3\sin^2\theta}{(MN^2)^{\frac{7}{2}}}\,\eta\psi_i(\eta)\,d\theta\,d\eta.$$

Hence $$\psi(x,y) = \frac{15}{\pi}y^2\int_{\eta=0}^{\infty}\int_{\theta=0}^{\pi}\frac{x^3\sin^2\theta}{(MN^2)^{\frac{7}{2}}}\,\eta\psi_i(\eta)\,d\theta\,d\eta - \frac{3}{\pi}y^2\int_{\eta=0}^{\infty}\int_{\theta=0}^{\pi}\frac{x^2\sin^2\theta}{(MN^2)^{\frac{5}{2}}}\,\eta^2 v_i(\eta)\,d\theta\,d\eta. \tag{5.5}$$

5.2. *Velocity field*

u and v are easily obtained through equations (2.5) and (5.5); of particular interest is the velocity on the axis $u_0(x)$, which can be obtained as the limit of $2\psi/y^2$ as $y \to 0$. After an integration by parts, we get

$$u_0(x) = 3x^3\int_0^{\infty}\frac{\eta u_i(\eta)}{(x^2+\eta^2)^{\frac{5}{2}}}\,d\eta - 3x^2\int_0^{\infty}\frac{\eta^2 v_i(\eta)}{(x^2+\eta^2)^{\frac{5}{2}}}\,d\eta. \tag{5.6}$$

Another integration by parts yields

$$u_0(x) = x^3\int_0^\infty \frac{du_i/d\eta}{(x^2+\eta^2)^{\frac{3}{2}}}\,d\eta - x^2\int_0^\infty \frac{d[\eta v_i(\eta)]/d\eta}{(x^2+\eta^2)^{\frac{3}{2}}}\,d\eta + u_i(0). \tag{5.7†}$$

5.3. *Pressure field*

Equation (3.17) becomes

$$\bar{p}-\bar{p}_\infty = -\frac{2}{y}\frac{\partial}{\partial y}(\partial V_1/\partial x + V_2), \tag{5.8}$$

where V_1 and V_2 are given by equations (5.4). We only give the results in final form for the pressure on the x-axis and for the base pressure.

On the x-axis

$$\bar{p}_0(x)-\bar{p}_\infty = -2\int_0^\infty \frac{\eta^2 du_i(\eta)/d\eta}{(x^2+\eta^2)^{\frac{3}{2}}}\,d\eta - 2x\int_0^\infty \frac{d[\eta v_i(\eta)]/d\eta}{(x^2+\eta^2)^{\frac{3}{2}}}\,d\eta. \tag{5.9}$$

In the base-flow problem, and for the same reasons as in two-dimensional flow, we also obtain the pressure on the base ($x = 0, 0 \leqslant y \leqslant 1$) directly from equations (5.4), although this is not possible for $y \geqslant 1$:

$$\bar{p}_b(y)-\bar{p}_\infty = \frac{3}{\pi}\int_{\eta=1}^\infty\int_{\theta=0}^\pi \frac{\sin^2\theta\,\eta\psi_i(\eta)}{(MN^2)^{\frac{5}{2}}}\,d\theta\,d\eta$$
$$+\frac{15}{\pi}y\int_{\eta=1}^\infty\int_{\theta=0}^\pi \frac{y^2-\eta^2}{(MN^2)^{\frac{7}{2}}}\sin^2\theta\,\eta\psi_i(\eta)\,d\theta\,d\eta, \tag{5.10}$$

where now $\qquad MN^2 = y^2+\eta^2-2y\eta\cos\theta.$

However, a simpler expression, from the point of view of numerical applications, can be obtained in a different way. We write

$$\frac{\partial V_1}{\partial x} = \int_\infty^x \frac{\partial^2 V_1}{\partial x^2}\,dx,$$

and use the following properties of the function $\partial^2 V_1/\partial x^2$:

$$D^2\left(\frac{\partial^2 V_1}{\partial x^2}\right) = 0,$$

$$\left.\frac{\partial^2 V_1}{\partial x^2}\right|_{x=0} = -\frac{d^2\psi_i}{dy^2}+\frac{1}{y}\frac{d\psi_i}{dy} = -y\frac{du_i}{dy}.$$

Applying now equation (5.3), we see that

$$\frac{\partial^2 V_1}{\partial x^2} = -\frac{3}{\pi}y^2\int_{\eta=0}^\infty\int_{\theta=0}^\pi \frac{x\sin^2\theta}{(MN^2)^{\frac{5}{2}}}\eta^2\frac{du_i}{d\eta}\,d\theta\,d\eta,$$

where $\qquad MN^2 = x^2+y^2+\eta^2-2y\eta\cos\theta.$

The integration with respect to x, from ∞ to x, yields

$$\frac{\partial V_1}{\partial x} = \frac{1}{\pi}y^2\int_{\eta=0}^\infty\int_{\theta=0}^\pi \frac{\sin^2\theta}{(MN^2)^{\frac{3}{2}}}\eta^2\frac{du_i}{d\eta}\,d\theta\,d\eta.$$

† Equation (5.7) assumes that both $u_i(\eta)$ and $v_i(\eta)$ are continuous functions.

After differentiating with respect to y, and making some transformations, we obtain

$$\bar{p}_b(y)-\bar{p}_\infty = -\frac{1}{\pi}\int_{\eta=1}^{\infty}\int_{\theta=0}^{\pi}\frac{\sin^2\theta}{(MN^2)^{\frac{3}{2}}}\eta^2\frac{du_i}{d\eta}\,d\theta\,d\eta - \frac{3}{\pi}\int_{\eta=1}^{\infty}\int_{\theta=0}^{\pi}\frac{\sin^2\theta}{(MN^2)^{\frac{5}{2}}}(\eta^2-y^2)\,\eta^2\frac{du_i}{d\eta}\,d\theta\,d\eta, \tag{5.11}$$

where now $$MN^2 = y^2+\eta^2-2y\eta\cos\theta.$$

The main advantage of equation (5.11) over (5.10) lies in the substitution of $\eta(du_i/d\eta)$ for ψ_i in the integrand. Equation (5.11) is the counterpart of (4.10) in two-dimensional flow.

6. Numerical examples and discussions

6.1. *Streamline patterns*

The streamline patterns in two-dimensional (figures 3 and 4) and axisymmetric flows (figures 5 and 6) were calculated using the following boundary conditions at $x = 0$:

$$u_i(y) = \begin{cases} 0 & (0\leqslant y\leqslant 1), \\ \tfrac{1}{2}[1-\cos\{\pi(y-1)/\delta\}] & (1\leqslant y\leqslant \delta+1), \\ 1 & (y\geqslant \delta+1), \end{cases} \tag{6.1}$$

where $u_i(-y) = u_i(y)$ in the two-dimensional case, and

$$v_i(y) = \begin{cases} 0 & (0\leqslant y\leqslant 1), \\ \tfrac{1}{2}v_M[1-\cos\{\pi(y-1)/\delta\}] & (1\leqslant y\leqslant 2\delta+1), \\ 0 & (y\geqslant 2\delta+1), \end{cases} \tag{6.2}$$

where $v_i(-y) = -v_i(y)$ in the two-dimensional case. δ was taken equal to 1; u_i and v_i are shown in figure 2. The choice of cosine profiles is arbitrary except for one condition which will be discussed later in relation to the base pressure.† The calculations were made on an IBM 7090 computer, and the integrations were performed according to Simpson's rule. Some particular problems which arise in the axisymmetric case in the calculation of double integrals are discussed in Appendix II. The terms contributed to the stream function by u_i and v_i (with $v_M = 1$) were calculated separately. The stream function corresponding to the combined boundary conditions (6.1) and (6.2) for any v_M could then be obtained by a simple linear combination. The two cases considered in figures 3 to 6 are

$$v_M = 0, \quad \text{i.e. } v_i \equiv 0,$$

and $$v_M = 0{\cdot}2.$$

They differ mainly by the existence, in the latter case, of a recirculating flow region behind the base, extending only a fraction of the base dimension down-

† Calculations were first carried out for axisymmetric flows. It was noticed afterwards that the integrations occurring in the two-dimensional case can be performed analytically when u_i and v_i are polynomials in y; however, it was decided to keep the boundary conditions (6.1) and (6.2) so that comparison could be made between the two-dimensional and axisymmetric flows.

stream. The volume flow rate involved in this zone is extremely small ($\sim 10^{-4}$), and so is the velocity. The existence of a recirculating flow is related to the sign of $u_0(x)$, the velocity on the x-axis, equations (4.7) and (5.6). The first term in u_0 is always positive for $x > 0$ (assuming $u_i(\eta) \geqslant 0$), and goes to zero with x like x^3.

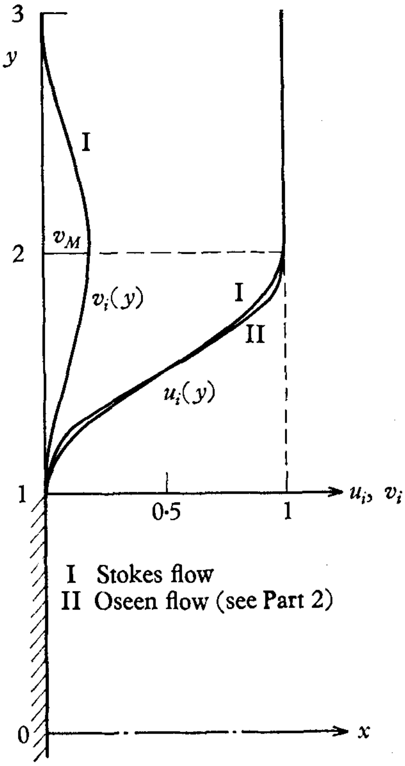

FIGURE 2. Velocity profiles at $x = 0$, $\delta = 1$.

The second term† is negative if $v_i(\eta) \geqslant 0$, positive if $v_i(\eta) \leqslant 0$, and goes to zero with x like x^2. Therefore, if $v_i(\eta) \geqslant 0$, $u_0(x)$ will be negative for x small enough, and there will be a rear stagnation point and a recirculating flow region; if $v_i(\eta) \leqslant 0$ or if $v_i(\eta) \equiv 0$, there is no such region. Furthermore, it is easily seen that the larger v_M is, the farther downstream will the rear stagnation point lie. For a value of $v_M = 0{\cdot}2$, which may be considered as large, it is only at a distance smaller than $0{\cdot}3a_1$ behind the base.

6.2. *Velocity and pressure on x-axis*

The velocity and pressure on the x-axis, corresponding to the boundary conditions (6.1) and (6.2) are shown in figures 7 and 8 for $\delta = 1$, and in figure 9 for $\delta = 10$. The influence of $v_i(\eta)$ ($\delta = 1$) is seen to be very small for the values of v_M

† The term can be written in the form

$$-\frac{4}{\pi}x^2\int_1^\infty \frac{\eta v_i(\eta)}{(x^2+\eta^2)^2}\,d\eta$$

in the two-dimensional case.

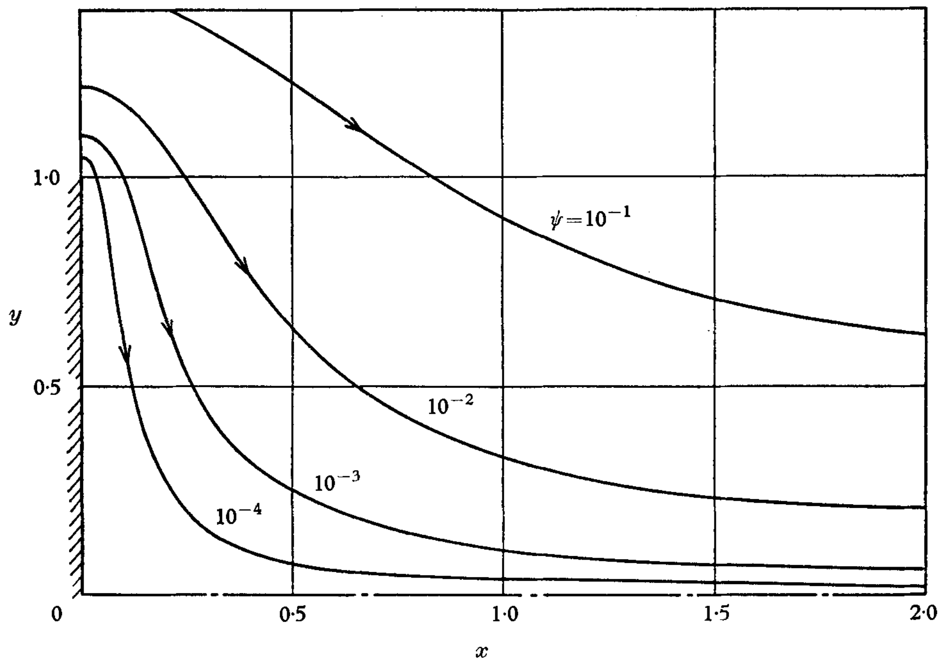

FIGURE 3. Streamline pattern in two-dimensional Stokes flow. $\delta = 1$, $v_i \equiv 0$.

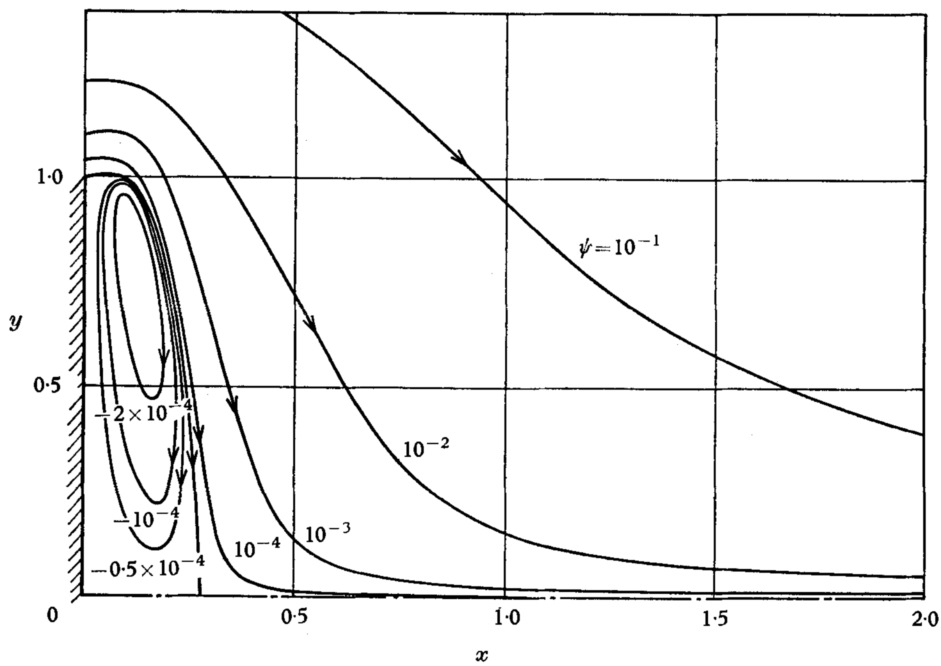

FIGURE 4. Streamline pattern in two-dimensional Stokes flow. $\delta = 1$, $v_M = 0{\cdot}2$.

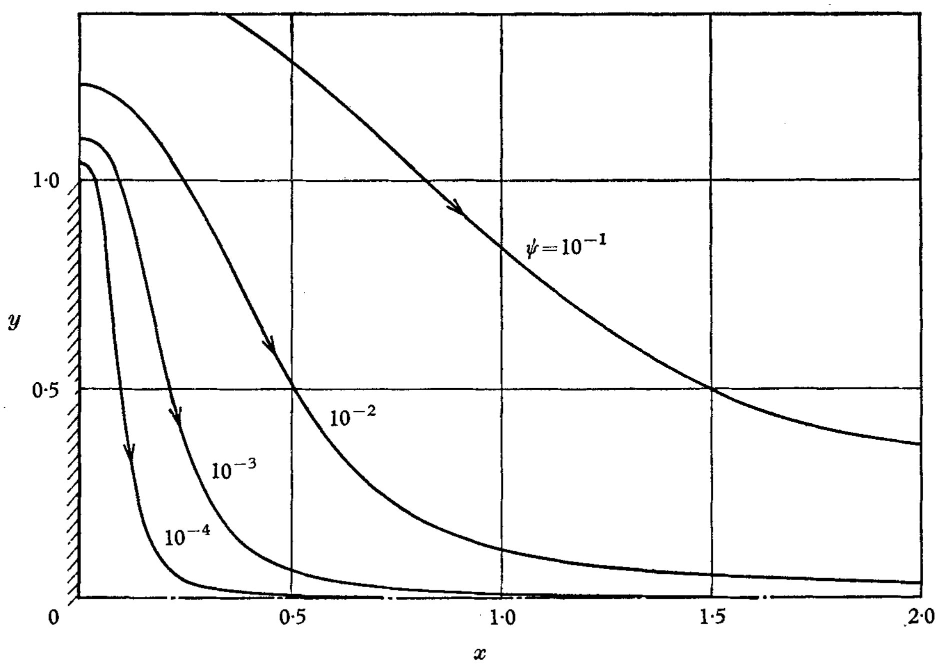

FIGURE 5. Streamline pattern in axisymmetric Stokes flow. $\delta = 1$, $v_i \equiv 0$.

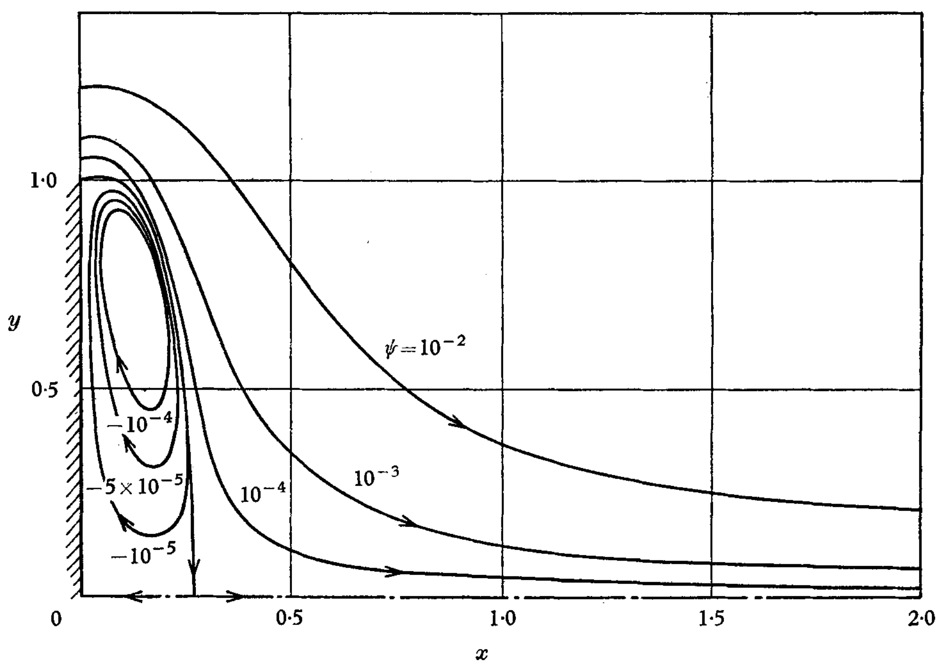

FIGURE 6. Streamline pattern in axisymmetric Stokes flow. $\delta = 1$, $v_M = 0{\cdot}2$.

considered; in particular, for $v_M = 0{\cdot}2$, and for small values of x, the scales for u_0 and $\bar{p}_0 - \bar{p}_\infty$ must be greatly enlarged in order to display the existence of negative velocities and of the minima which occur at a small positive value of x. The main differences between the two-dimensional and axisymmetric cases consist in a more rapid rise for $u_0(x)$ and lower pressures in the latter case. The

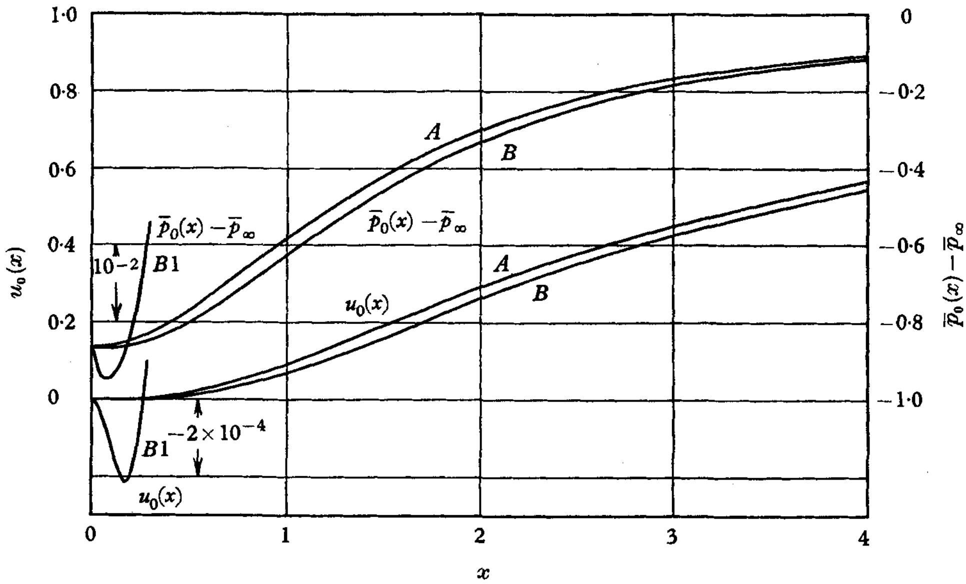

FIGURE 7. Velocity and pressure on x-axis in two-dimensional Stokes flow. $\delta = 1$. A, $v_t \equiv 0$; B, $v_M = 0{\cdot}2$; $B1$, as B with enlarged ordinate.

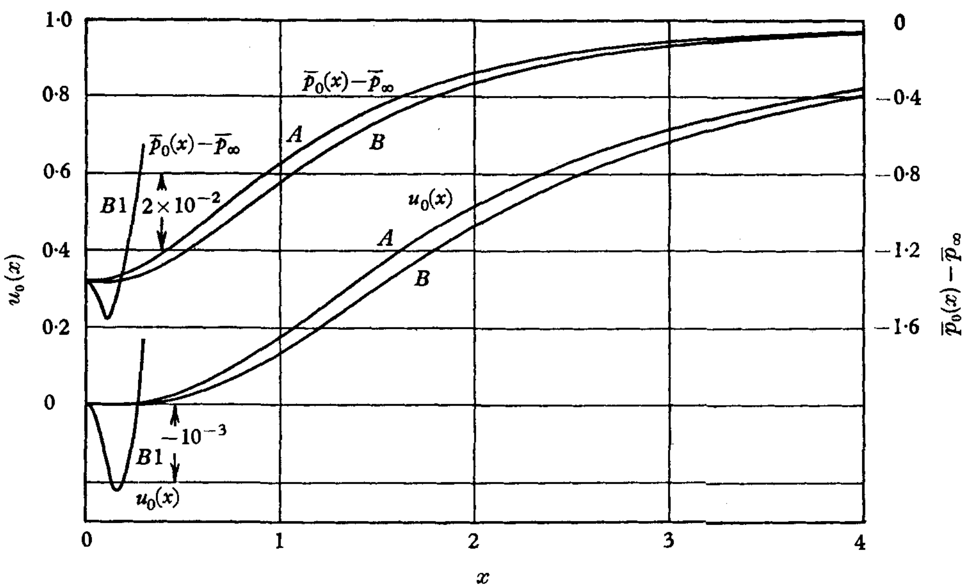

FIGURE 8. Velocity and pressure on x-axis in axisymmetric Stokes flow. $\delta = 1$. A, $v_t \equiv 0$; B, $v_M = 0{\cdot}2$; $B1$, as B with enlarged ordinate.

rise in velocity is slower for $\delta = 10$ than for $\delta = 1$ if it is measured versus x, but is actually more rapid if measured versus x/δ; pressure variations are smaller.

As mentioned in the introduction, local validity of the Stokes approximation can be examined *a posteriori* by comparing inertia to viscous (or pressure) terms as given by the solution of the Stokes equation. This was done on the x-axis only,

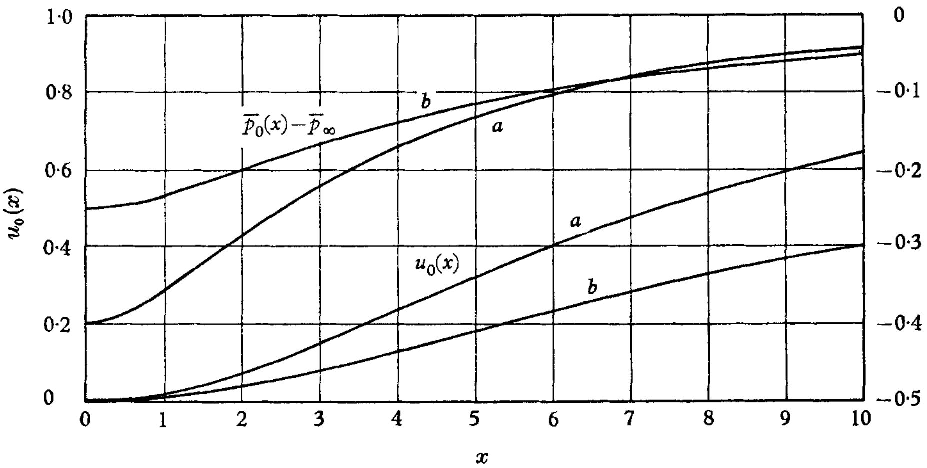

FIGURE 9. Velocity and pressure on x-axis in Stokes flow. $\delta = 10$, $v_i \equiv 0$. a, Axisymmetric flow; b, two-dimensional flow.

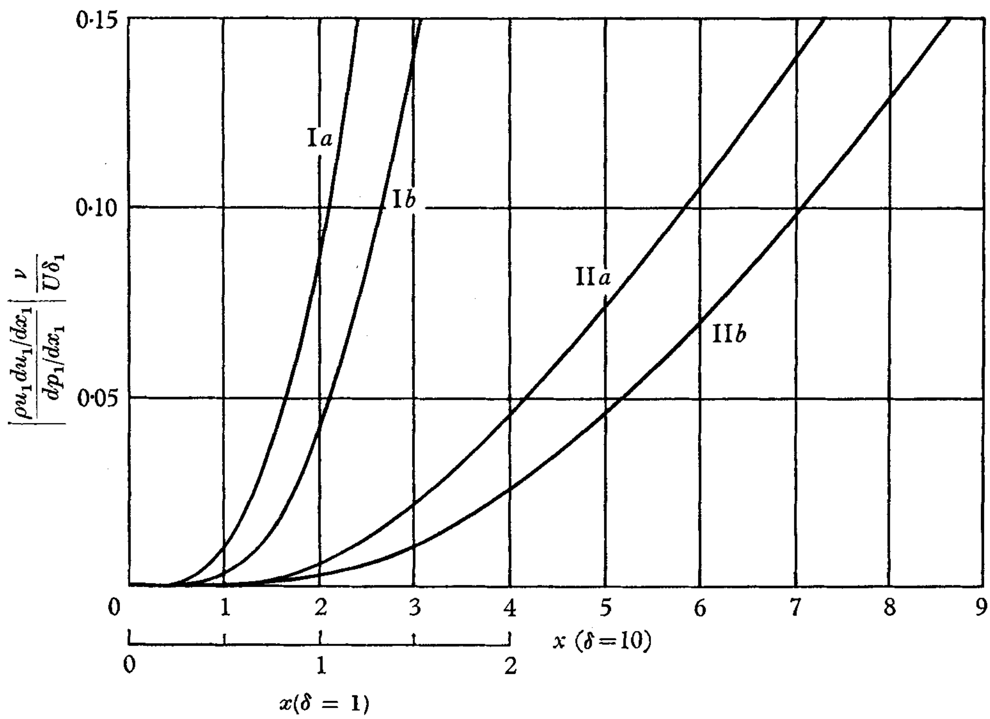

FIGURE 10. Ratio of inertia to pressure terms on x-axis in Stokes flow. I, $\delta = 1$; II, $\delta = 10$; a, axisymmetric flow; b, two-dimensional flow.

using the results of figures 7, 8, and 9. Velocity and pressure gradients were actually measured on these curves, the resulting accuracy being sufficient for our purpose. The results are shown in figure 10, and are summarized in the table 1 giving the conditions for the ratio of inertia to pressure forces to be less than 0·1, this value being considered small enough for the Stokes approximation to apply.

		$U\delta_1/\nu$ = 1	$U\delta_1/\nu$ = 10
Two-dimensional	$\delta = 1$	$x < 1{\cdot}3$	$x < 0{\cdot}7$
	$\delta = 10$	$x < 7{\cdot}0$	$x < 2{\cdot}9$
Axisymmetric	$\delta = 1$	$x < 1{\cdot}0$	$x < 0{\cdot}5$
	$\delta = 10$	$x < 5{\cdot}8$	$x < 2{\cdot}3$

TABLE 1

6.3. *Base pressure*

From equations (4.10) and (5.11) we note that the base pressure depends only on $u_i(\eta)$.

The base pressure, in the vicinity of the corner $y = 1$, is very sensitive to the shape of the initial profile $u_i(y)$ for $y \to 1$; in particular, if $(du_i/dy)(1+) \neq 0$, the pressure becomes logarithmically infinite at $y = 1$. This can be seen directly from equation (4.10) in the two-dimensional case; in the axisymmetric case one must use the fact (see Appendix II) that $I_{\frac{3}{2}}(\gamma) \sim (\gamma - 1)^{-2}$ as $\gamma \to 1+$, to show that the second term in equation (5.10) goes to infinity.

Physically this requirement that the initial profile must be of separation type arises because the initial profile becomes a free shear layer as soon as it leaves the body; if $(du_i/dy)(+1) \neq 0$, a discontinuity in shear will be introduced at the corner. The consequence of this discontinuity is the infinite negative pressure at the corner. The initial profile, equation (6.1), was specifically chosen to be of separation type to avoid this difficulty.

Base pressure distributions are shown in figure 11 for $\delta = 1$ and in figure 12 for $\delta = 10$. For $\delta = 1$ there is an appreciable decrease of pressure in the lateral (or radial) direction, this feature being still more accentuated in the axisymmetric case. This result can be compared with base pressure measurements at low Reynolds numbers (although outside the range of the present investigation) by Kavanau (1956) exhibiting similar pressure variation with y, and it can be contrasted with the assumption usually made at high Reynolds numbers of uniform pressure in the base region.

6.4. *General comments*

It is interesting to note that the Stokes paradox does not apply to the solution for two-dimensional flow given in § 4, since the condition for uniform flow at infinity is satisfied. This is to be compared with a general theorem by Finn & Noll (1957) according to which the only two-dimensional Stokes flow about a body, with bounded velocity at infinity, is the state of rest; the existence of a solution in our case is therefore related to the absence of solid surfaces in the half-space $x > 0$.

After this work was completed, it came to our attention that the general solution in two-dimensional flow, with u and v given as boundary conditions at

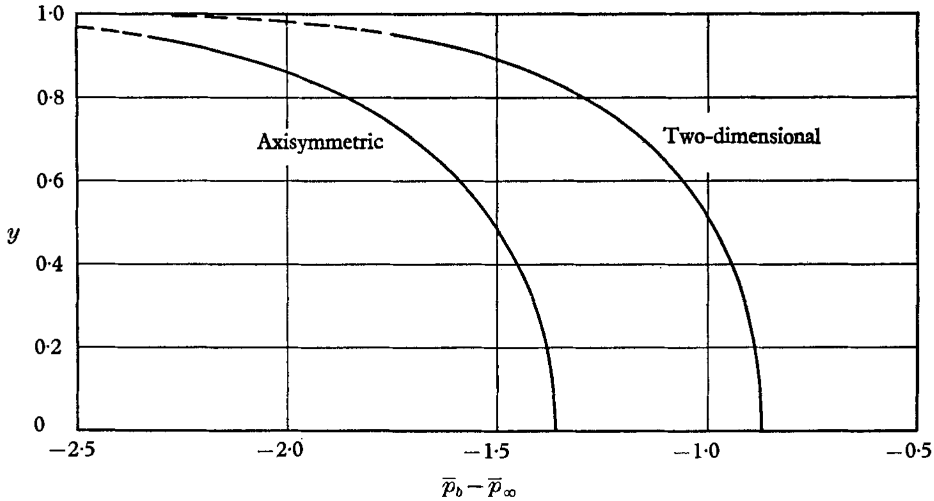

FIGURE 11. Base pressure. $\delta = 1$. Stokes flow.

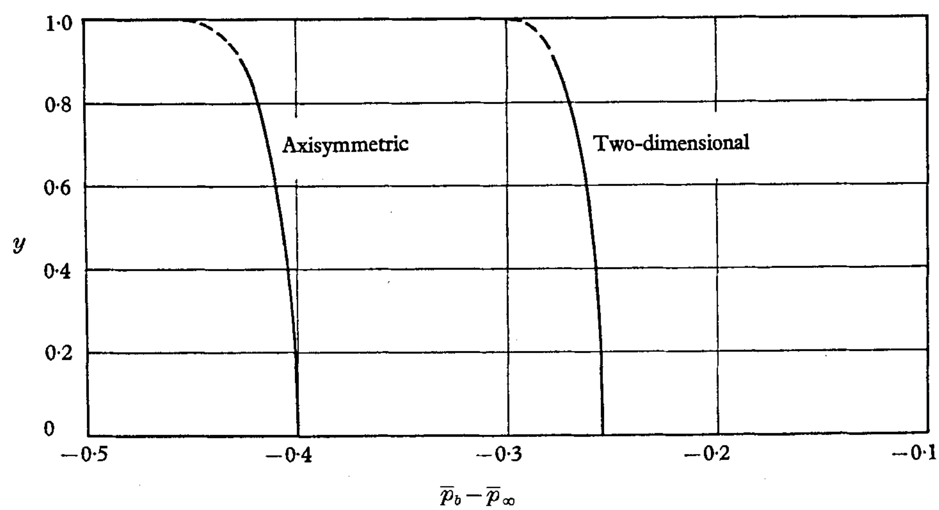

FIGURE 12. Base pressure. $\delta = 10$. Stokes flow.

$x = 0$, had been obtained in a different form by Förste (1963) and applied to a particular jet problem. Particular solutions of the jet problem have also been given by Dean (1936).

As already noted, the integrations occurring in the solution for two-dimensional flow can be carried out analytically when u_i and v_i are polynomials in y; the pressure at $x = 0$ can then be obtained without difficulty.

In the particular case where $v_i \equiv 0$, a simple relation exists between $u_i(y)$ and $\Omega_i(y) = \Omega(0, y)$, the vorticity in the base plane; indeed, if $v_i \equiv 0$, $V_2 \equiv 0$, and from equation (3.16) we obtain

$$\Omega_i(y) = -2\frac{du_i}{dy},$$

and as a consequence

$$\left.\frac{\partial v}{\partial x}\right|_{x=0} = -\frac{du_i}{dy}.$$

Therefore, viscous forces exerted on the plane $x = 0$ are zero for all y, and vorticity is zero on the base. In two-dimensional flow these results hold also if $v_i(y)$ is a constant not necessarily zero, since then $V_2(x, y) = \text{const.} = v_i$.

Appendix I

The purpose of this appendix is to derive the solution of the following boundary-value problem

$$D^2 V = 0, \quad \text{where} \quad D^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y}, \tag{A 1}$$

$$V(0, y) = V_i(y) \quad (y \geqslant 0), \tag{A 2a}$$

$$V(x, 0) = 0 \qquad (x \geqslant 0), \tag{A 2b}$$

in the quarter plane $x > 0$, $y > 0$.

Let

$$\phi = V/y^2. \tag{A 3}$$

We shall verify *a posteriori* that $\phi(x, 0)$ is finite so that condition (A 2b) is satisfied. We find

$$D^2 V = y^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{3}{y}\frac{\partial\phi}{\partial y}\right)$$

so that ϕ must be the solution of

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{3}{y}\frac{\partial\phi}{\partial y} = 0, \tag{A 4}$$

$$\phi(0, y) = \phi_i(y) = \frac{1}{y^2}V_i(y). \tag{A 5}$$

Consider now a five-dimensional space with co-ordinates x, x_1, x_2, x_3, x_4 such that $y^2 = \sum_{i=1}^{4} x_i^2$, and the function $\overline{\phi}(x, x_1, x_2, x_3, x_4)$ defined by $\overline{\phi} = \phi(x, y)$; it is easily seen that

$$\frac{\partial^2\overline{\phi}}{\partial x^2} + \sum_{i=1}^{4}\frac{\partial^2\overline{\phi}}{\partial x_i^2} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{3}{y}\frac{\partial\phi}{\partial y} = 0$$

so that $\overline{\phi}$ is a harmonic function in these five variables; since $\overline{\phi}$ is known for $x = 0$ (being equal to $\phi_i(y)$), we can write directly, using Poisson's integral in a five-dimensional space and for the domain $x > 0$ (see Courant & Hilbert 1962, vol. II, p. 268).

$$\overline{\phi} = \frac{3x}{4\pi^2}\iiiint\limits_{\substack{\xi_i=-\infty \\ i=1,\ldots,4}}^{+\infty} \frac{\phi_i(\eta)\, d\xi_1\, d\xi_2\, d\xi_3\, d\xi_4}{[x^2 + (x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2 + (x_4-\xi_4)^2]^{\frac{5}{2}}}, \tag{A 6}$$

where

$$\eta^2 = \sum_{i=1}^{4}\xi_i^2.$$

Now replace $\xi_i\,(i=1,\ldots,4)$ by polar co-ordinates η, θ_1, θ_2, θ_3 through the transformation formulas

$$\xi_1 = \eta\cos\theta_1,$$
$$\xi_2 = \eta\sin\theta_1\cos\theta_2,$$
$$\xi_3 = \eta\sin\theta_1\sin\theta_2\cos\theta_3,$$
$$\xi_4 = \eta\sin\theta_1\sin\theta_2\sin\theta_3.$$

One finds

$$\frac{\partial(\xi_1,\xi_2,\xi_3,\xi_4)}{\partial(\eta,\theta_1,\theta_2,\theta_3)} = \eta^3\sin^2\theta_1\sin\theta_2,$$

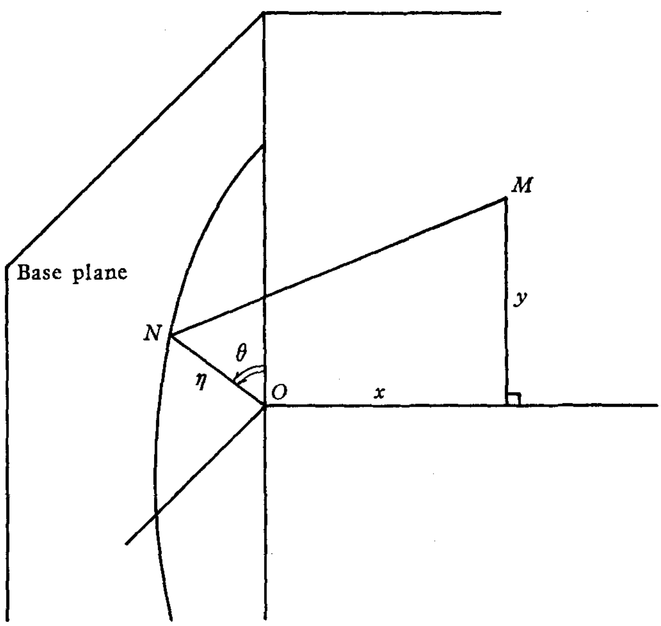

FIGURE 13.

so that equation (A 6) becomes

$$\bar{\phi}(x,x_1,x_2,x_3,x_4) = \frac{3x}{4\pi^2}\int_{\eta=0}^{\infty}\underset{\substack{\theta_i=0\\(i=1,2,3)}}{\int^{\pi}\int^{\pi}\int^{2\pi}}\frac{\phi_1(\eta)\,\eta^3\sin^2\theta_1\sin\theta_2\,d\eta\,d\theta_1\,d\theta_2\,d\theta_3}{D^{\frac{5}{2}}}, \tag{A 7}$$

where

$$D = x^2+y^2+\eta^2-2\eta(x_1\cos\theta_1-x_2\sin\theta_1\cos\theta_2-x_3\sin\theta_1\sin\theta_2\cos\theta_3$$
$$-x_4\sin\theta_1\sin\theta_2\sin\theta_3).$$

By definition, $\bar{\phi}$ is a function of x_1, x_2, x_3, x_4 only through the combination $y^2 = \sum_{i=1}^{4} x_i^2$; in particular, if we consider the point $(x, x_1 = y, x_2 = x_3 = x_4 = 0)$, we get

$$D = x^2+y^2+\eta^2-2y\eta\cos\theta_1.$$

Performing the integration with respect to θ_2 and θ_3 in equation (A 7), and dropping the subscript 1 in θ_1, we finally obtain

$$\phi(x,y) = \frac{3}{\pi}x\int_{\eta=0}^{\infty}\int_{\theta=0}^{\pi}\frac{\phi_i(\eta)\,\eta^3\sin^2\theta\,d\eta\,d\theta}{[x^2+y^2+\eta^2-2y\eta\cos\theta]^{\frac{5}{2}}}. \tag{A 8}$$

From this expression one can easily show that $\phi(x,0)$ is finite. Going back to $V(x,y)$ (equations (A 3) and (A 5)) we find

$$V(x,y) = \frac{3}{\pi} y^2 x \int_{\eta=0}^{\infty} \int_{\theta=0}^{\pi} \frac{V_i(\eta)\,\eta \sin^2\theta\, d\eta\, d\theta}{[x^2+y^2+\eta^2-2y\eta\cos\theta]^{\frac{5}{2}}}. \tag{A 9}$$

The denominator in the integrand can be given a geometrical interpretation in the physical space; let M be the point (x,y) and N the point of the plane $x=0$ with polar co-ordinates (η,θ) (see figure 13). Then $MN^2 = x^2+y^2+\eta^2-2y\eta\cos\theta$.

The quantity $\eta\, d\eta\, d\theta$ can also be interpreted as the surface element in the plane $x=0$ so that the double integral in (A 9) is a surface integral over half of the plane $x=0$.

Equation (A 9) represents the solution as long as the integral has a meaning. Brousse (1956) has shown this to be the case when $V_i(\eta) \sim \eta^\alpha$ as $\eta \to \infty$, with $\alpha < 3$.

Appendix II

The determination of the stream function and of the base pressure in axisymmetric flow involves the calculation of double integrals of the form

$$J_m = \frac{1}{\pi} \int_{\eta=0}^{\infty} \int_{\theta=0}^{\pi} \frac{f(\eta)\sin^2\theta}{(MN^2)^{\frac{1}{2}m}}\, d\theta\, d\eta,$$

where
$$MN^2 = x^2+y^2+\eta^2-2y\eta\cos\theta,$$

and where $m = 3$, 5, or 7.

In this appendix, we carry out the integration with respect to θ by expanding the integrand in series; the remaining integration is to be performed numerically.

Let
$$\gamma = \frac{x^2+y^2+\eta^2}{2y\eta} = 1 + \frac{x^2+(y-\eta)^2}{2y\eta}.$$

Note that $\gamma > 1$, if $x \neq 0$ (y and $\eta > 0$).

Then
$$MN^2 = (x^2+y^2+\eta^2)\left(1-\frac{\cos\theta}{\gamma}\right).$$

Substituting this expression for MN^2 into J_m, we get

$$J_m = \int_{\eta=0}^{\infty} \frac{f(\eta)}{(x^2+y^2+\eta^2)^{\frac{1}{2}m}} I_{\frac{1}{2}m}(\gamma)\, d\eta,$$

where
$$I_{\frac{1}{2}m} = \frac{1}{\pi} \int_{\theta=0}^{\pi} \frac{\sin^2\theta\, d\theta}{\left(1-\dfrac{\cos\theta}{\gamma}\right)^{\frac{1}{2}m}}.$$

Calculation of $I_{\frac{5}{2}}$

Since $\cos\theta/\gamma < 1$, we can expand the integrand in series

$$\left(1-\frac{\cos\theta}{\gamma}\right)^{-\frac{5}{2}} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(2n+3)!}{n!\,(n+1)!\,2^{2n+1}} \left(\frac{\cos\theta}{\gamma}\right)^n.$$

Integrating term by term we find that

$$\int_0^\pi \sin^2\theta \cos^{2n+1}\theta\, d\theta = 0,$$

$$\int_0^\pi \sin^2\theta \cos^{2n}\theta\, d\theta = \pi \frac{(2n)!}{(n+1)!\, n!\, 2^{2n+1}}.$$

Hence

$$I_{\frac{5}{2}} = \frac{1}{3}\sum_{n=0}^{\infty} C_{2n}\left(\frac{1}{\gamma^2}\right)^n,$$

where

$$C_{2n} = \frac{1}{2^{6n+2}} \frac{(4n+3)!}{n!(n+1)!\,(2n+1)!}.$$

The C_{2n}'s were calculated using the recurrence formula

$$\frac{C_{2n+2}}{C_{2n}} = 1 + \frac{3}{16(n+1)\,(n+2)}.$$

They form an increasing sequence, with $C_0 = \frac{3}{2}$ and

$$C_{2n} \sim C_\infty\left[1 - \frac{3}{16n} + O\left(\frac{1}{n^2}\right)\right] \quad \text{as} \quad n \to \infty,$$

where

$$C_\infty = \frac{4\sqrt{2}}{\pi} = 1{\cdot}8006....$$

Calculation of $I_{\frac{7}{2}}$

$I_{\frac{7}{2}}(\gamma)$ is easily obtained from $I_{\frac{5}{2}}$ if we notice that

$$I_{\frac{7}{2}} = I_{\frac{5}{2}} - \tfrac{2}{5}\gamma(d/d\gamma)\,(I_{\frac{5}{2}}).$$

The result can be written in the form

$$3I_{\frac{7}{2}} = \tfrac{4}{5}C_\infty \frac{\gamma^{-2}}{(\gamma^{-2}-1)^2} + \sum_{n=0}^{\infty} B_{2n}\,\gamma^{-2n},$$

where

$$B_{2n} = C_{2n} - \tfrac{4}{5}n(C_\infty - C_{2n}).$$

The coefficients B_{2n} form an increasing sequence with $B_0 = \frac{3}{2}$ and

$$\lim_{n\to\infty} B_{2n} = B_\infty = \tfrac{17}{20}C_\infty = 1{\cdot}530....$$

Calculation of $I_{\frac{3}{2}}$†

Following the method used for $I_{\frac{5}{2}}$, and using

$$\left(1 - \frac{\cos\theta}{\gamma}\right)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} \frac{(2n+1)!}{2^{2n}(n!)^2}\left(\frac{\cos\theta}{\gamma}\right)^n,$$

we obtain

$$I_{\frac{3}{2}} = \sum_{n=0}^{\infty} H_{2n}\,\gamma^{-2n},$$

† $I_{\frac{3}{2}}$ is simply related to the Legendre function of the second kind and order $\frac{1}{2}$:

$$Q_{\frac{1}{2}}(\gamma) = \frac{1}{2\sqrt{2}}\int_0^\pi \frac{\sin^2\theta\, d\theta}{(\gamma - \cos\theta)^{\frac{3}{2}}} = \pi(2\gamma)^{-\frac{3}{2}}\, I_{\frac{3}{2}}(\gamma).$$

where
$$H_{2n} = \frac{1}{2^{6n+1}}\frac{(4n+1)!}{n!\,(n+1)!\,(2n)!} = \frac{C_{2n}}{4n+3}.$$

Note that
$$H_{2n} \sim \frac{C_\infty}{4n} = \frac{\sqrt{2}}{\pi}\frac{1}{n} \quad \text{as} \quad n \to \infty.$$

To reduce the minimum number of terms in the series which must be retained for a given accuracy, we write $I_{\frac{3}{2}}$ in the form

$$I_{\frac{3}{2}} = H_0 + \tfrac{1}{4}C_\infty \sum_{n=1}^{\infty}\frac{\gamma^{-2n}}{n} + \sum_{n=1}^{\infty}(H_{2n} - \tfrac{1}{4}C_\infty/n)\gamma^{-2n},$$

or
$$I_{\frac{3}{2}} = \frac{1}{2} - \frac{\sqrt{2}}{\pi}\log_e(1-\gamma^{-2}) - \sum_{n=1}^{\infty}A_{2n}\gamma^{-2n},$$

where
$$A_{2n} = \tfrac{1}{4}C_\infty/n - H_{2n} = \frac{3C_\infty + 4n(C_\infty - C_{2n})}{4n(4n+3)} > 0.$$

The A_{2n}'s form a decreasing sequence with

$$A_{2n} \approx \frac{15}{16}\frac{\sqrt{2}}{\pi}\frac{1}{n^2} \quad \text{as} \quad n \to \infty.$$

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